DHANAMANJURI UNIVERSITY

Examination- 2024 (June)
M.Sc. 2nd Semester

Name of Programme	**:**	**M.Sc. Mathematics**
Paper Type	**:**	**Theory**
Paper Code	**:**	**MAT-507**
Paper Title	**:**	**Real Analysis-II**

Full Marks : 40

Pass Marks : 16 **Duration: 2 Hours**

The figures in the margin indicate full marks for the questions.

Answer any four from the following questions: **10×4=40**

1. Let $\{E_n\}$ be a countable collection of sets. Then prove that $m^*(\bigcup_n E_n) \leq \sum_n m^*(E_n)$, where $m^*(E_n)$ denotes the Lebesgue outer measure of set E_n. Further, if E is an accountable set, then show that $m^*(E) = 0$. Compute the Lebesgue outer measure of $A = [-1, 1] \cup [2, 5]$. 5 + 3 + 2 = 10

2. Define Borel set. Show that every Borel set in $\mathbb{R}$ is a measurable set. In particular, show that each open set and closed set is measurable. 1 + 7 + 2 = 10

3. Define generalized Lebesgue integral. Let f_n be a sequence of non-negative measurable functions and $f_n \to f$ a.e. on E. Then prove that
$$\int_E f \leq \lim_{n\to\infty} \inf \int_E f_n.$$
2 + 8 = 10

4. a) If f is of bounded variation on $[a, b]$, then $T_a^b = P_a^b + N_a^b$ and $f(b) - f(a) = P_a^b - N_a^b$.

   b) Calculate the four Dini derivatives at $x = 0$ of the following
$$f(x) = \begin{cases} ax\sin^2\frac{1}{x} + bx\cos^2\frac{1}{x}, & x > 0 \\ 0, & x = 0 \\ a'x\sin^2\frac{1}{x} + b'x\cos^2\frac{1}{x}, & x < 0 \end{cases}$$
   where $a < a',\ b < b'$. 5 + 5 = 10

5. Prove that function f defined on $[a, b]$ is of bounded variation if and only if it can be expressed as a difference of two monotone increasing real valued functions on $[a, b]$. If f is of bounded variation, then show that f is differentiable a.e. on $[a, b]$. 8+2=10

6. Define a measurable function. Let f be a bounded and measurable function defined on $[a, b]$. If $F(x) = \int_a^x f(t)dt + F(a)$, then prove that $F'(x) = f(x)$ a.e. in $[a, b]$. 2+8=10

7. Prove that e^x is strictly convex on $\mathbb{R}$. Further, prove that a differentiable function ψ is convex on (a, b) if and only if ψ' is a monotone increasing function. $2 + 8 = 10$

8. Define conjugate number. Let $1 < p < \infty$, $1 < q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$, and let $f \in L^p(\mu)$ and $g \in L^q(\mu)$. Then prove that $fg \in L^1(\mu)$ and $\int |fg| \leq \left(\int |f|^p d\mu\right)^{1/p} \left(\int |g|^q d\mu\right)^{1/q}$. 2+8=10

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