DHANAMANJURI UNIVERSITY

Examination- 2024 (December)

M.Sc.1 st Semester

Name of Programme	**:**	**M.Sc. Mathematics**
Paper Type	**:**	**Theory**
Paper Code	**:**	**MAT-504**
Paper Title	**:**	**Complex Analysis-I**

Full Marks : 80

Pass Marks : 32 **Duration: 3 Hours**

The figures in the margin indicate full marks for the questions.

UNIT-1

Answer any three questions: **$10\times 3 = 30$**

1. a) Define continuity of a complex function at a plane. 1

   b) Let $f(z) = \begin{cases} \frac{1}{z}, & z \neq 0, \\ \infty, & z = 0, \\ 0, & z = \infty \end{cases}$. Prove that $f(z)$ is continuous in the extended complex plane. 9

2. Let $f(z)$ be defined in neighborhood of $z = a+ib$; u_x, u_y, v_x, v_y are continuous at (a, b) and satisfy $u_x = v_y, u_y = -v_x$. Then prove that $f'(x)$ exists at $z = a + ib$. 10

3. Let $u(x, y)$ be a function that is harmonic in a simple connected domain Ω and let $C : |S| = R$ be any circle contained in Ω. Then prove that $u(x, y)$ has integral representation
$u(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{R^2-r^2}{R^2-2Rrcos(t-\theta)+r^2} u(Re^{it})dt, 0 < r < R.$ 10

4. Let $f(z)$ be analytic in a domain Ω and let y be a simple closed contour in Ω, taken in positive sense. Then for all z interior to y prove that $f'(z) = \frac{1}{2\pi i} \int_y \frac{f(s)}{(s-z)^2} ds.$ 10

5. a) If a function is continuous on a domain D and $\int_y f(z)dz = 0$ for every closed contour y in D, then prove that $f(z)$ is analytic throughout D. 3

   b) Suppose $f(z)$ is analytic in a domain Ω and $C = \{z : |z - a| = R\}$ contained in Ω. Then prove that $|f^{(n)}(a)| \leq \frac{n!M_R}{R^n}, = 0, 1, 2, ...$ where $M_R = max_{z\in C}|f(z)|$. 3

   c) State and prove Liouville's Theorem. 4

UNIT-II

Answer any three questions: **10× 3 = 30**

6. a) Suppose that a function f is analytic in the angular region $R_1 < |z-a| < R_2$. Let C denote any positively oriented circle around a and lying in the domain. Then $f(z)$ can be expanded about any point in the annular domain
$R_1 < |z-a| < R_2$ as $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a_n)^n}$ where
$a_n = \frac{1}{2\pi i}\int_c \frac{f(s)}{(s-a)^{n+1}} ds, n = 0, 1, 2, ...$
and $b_n = \frac{1}{2\pi i}\int_c \frac{f(s)}{(s-a)^{-n+1}} ds, n = 1, 2, ..$ 6

b) Obtain the Laurent series expansion of $f(z) = \frac{z}{z^2-4z+3}$. 4

7. Define the following with an example: 10
   i) Singular Point
   ii) Isolated Singularity
   iii) Non-isolated Singularity
   iv) Removable singularity
   v) Pole

8. Find the singularities, their types and hence the corresponding Residues of the following functions: 10

   i) $\frac{z^2+16}{(z-1)^2(z+3)}$ ii) $\frac{\pi cot \pi z}{z^2}$.

9. Let $f(z)$ be analytic in $|z| < 1$ with a zero of order n at the origin. Suppose that $|f(z) \leq 1$ for all z in $|z| \leq 1$. Then $|f(z)| \leq |z|^n$ for $|z| < 1$ and $|f^{(n)}(0)| \leq n!$. The equality holds if $f(z) = cz^n, |c| = 1$. 10

10. State and prove Taylor Series Theorem for a Complex Function. 10

UNIT-III

Answer any three questions: **10× 2 = 20**

11. State and prove the necessary condition for conformality of a transformation. 10

12. Find all the bilinear transformations of the half plane Im $z \geq 0$ into the circle $|w| \leq 1$. 10

13. Show that the inversion map $w = \frac{1}{z}$ transforms circles and lines into circles and lines. 10

14. Find the fixed points and the normal form of the bilinear transformation $w = \frac{(2+i)z-2}{2-i}$. And classify their nature. 10

*****