DHANAMANJURI UNIVERSITY

Examination- 2024 (December)

M.Sc.1 st Semester

Name of Programme	:	**M.Sc. Mathematics**
Paper Type	:	**Theory**
Paper Code	:	**MAT-502**
Paper Title	:	**Real Analysis-I**

Full Marks : 80

Pass Marks : 32 **Duration: 3 Hours**

The figures in the margin indicate full marks for the questions.

Answer all the questions:

1. Answer any three of the following questions: **$10\times 3 = 30$**

a) i) If f is a bounded real valued function defined on $[a,b]$, α is monotonically increasing on $[a,b]$ and P^* is a refinement of P, then prove that $L(P,f,\alpha) \leq L(P^*,f,\alpha)$ and $U(P^*,f,\alpha) \leq U(P,f,\alpha)$.

ii) Let f be a bounded real valued function defined on $[a,b]$ and α be monotonically increasing on $[a,b]$. Prove that $\underline{\int_a^b} f\,d\alpha \leq \overline{\int_a^b} f\,d\alpha$.

b) i) Show that $f \in R(\alpha)$ on $[a,b]$ if and only if for every $\varepsilon > 0$, there exists a partition P of $[a,b]$ such that $U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$.

ii) Let f be a bounded real valued function defined on $[a,b]$. If $U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$ holds for some partition $P = \{a = x_0, x_1, \ldots, x_n = b\}$ of $[a,b]$ and some $\varepsilon > 0$, then show that $\sum_{i=1}^{n} |f(s_i) - f(t_i)|\Delta\alpha_i < \varepsilon$, if $s_i, t_i \in [x_{i-1}, x_i]$.

c) Let $f_1, f_2 \in \mathcal{R}(\alpha)$ on $[a,b]$. Show that $f_1 + f_2 \in \mathcal{R}(\alpha)$ on $[a,b]$ and $\int_a^b (f_1 + f_2)\,d\alpha = \int_a^b f_1\,d\alpha + \int_a^b f_2\,d\alpha$.

d) i) If f is continuous on $[a,b]$ and α is monotonically increasing on $[a,b]$, then show that $f \in \mathcal{R}(\alpha)$ on $[a,b]$.

ii) Let α be monotonically increasing on $[a,b]$. Suppose $f \in \mathcal{R}(\alpha)$ on $[a,b]$, $m \leq f \leq M, \phi$ is continuous on $[m,M]$ and $h(x) = \phi(f(x))$ on $[a,b]$. Show that $h \in \mathcal{R}(\alpha)$ on $[a,b]$.

e) i) Let $f \in \mathcal{R}$ on $[a,b]$ and let $F(x) = \int_a^x f(t)dt$ for all x in $[a,b]$. Show that F is continuous on $[a,b]$; and F is differentiable at $x_0 \in [a,b]$ and $F'(x_0) = f(x_0)$ if f is a continuous at x_0.

ii) If $f \in \mathcal{R}$ on $[a,b]$ and there is a differentiable function F on $[a,b]$ such that $F' = f$, then prove that $\int_a^b f(x)dx = F(b) - F(a)$.

2. Answer any three of the following questions: **10× 3 = 30**

a) Define the rearrangement of a sequence of real numbers. If $\sum_{n=1}^{\infty} a_n$ converges absolutely to A, the show that any rearrangement $\sum_{n=1}^{\infty} a'_n$ of $\sum_{n=1}^{\infty} a_n$ converges absolutely to$A$.

b) State and prove the Riemann's rearrangement theorem on series of real numbers.

c) i) Suppose $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in E$.
Put $M_n = \sup_{x\in E} |f_n(x) - f(x)|$. Prove that $f_n \to f$ uniformly on E and only if $M_n \to 0$ as $n \to \infty$.
ii) Show that the sequence $\{f_n\}$, where $f_n(x) = \frac{x}{1+nx^2}$, is uniformly convergent on any closed interval I.

d) i) Let $\{f_n\}$ be a sequence of continuous function defined on E, and let $f_n \to f$ uniformly on E. Prove that f is continuous on E.
ii) Let $\{f_n\}$ be a uniformly convergent sequence with uniform limit f on $[a, b]$ and let f_n be integrable on $[a, b]$ for all $n \in \mathbb{N}$. Prove that f is integrable on $[a, b]$ and $\int_a^b f(x)\,dx = \lim_{n\to\infty} \int_a^b f_n(x)\,dx$..

e) State and prove the Abel's test for uniform convergence of series of functions.

3. Answer any two of the following questions: **10 × 2= 20**

a) i) If $A \in L(\mathbb{R}^n, \mathbb{R}^m)$, then prove that $\|A\| < \infty$ and A is a uniformly continuous mapping of $\mathbb{R}^n$ into $\mathbb{R}^m$.
ii) If $A, B \in L(\mathbb{R}^n, \mathbb{R}^m)$ and c is a scalar, then prove that
$\|A + B\| \le \|A\| + \|B\|$ and $\|cA\| = |c|\|A\|$. Also show that $L(\mathbb{R}^n, \mathbb{R}^m)$ is a metric space with the distance between A and B defined by $\|A - B\|$.
iii)If $A \in L(\mathbb{R}^n, \mathbb{R}^m)$ and $B \in L(\mathbb{R}^m, \mathbb{R}^k)$, then prove that $\|BA\| \le \|B\|\,\|A\|$.

b) i) Let E be an open set in $\mathbb{R}^n$, f maps E into $\mathbb{R}^n$, $x \in E$ and $\lim_{h\to 0} \frac{|f(x+h)-f(x)-Ah|}{|h|} = 0$ holds with $A = A_1$ and with $A = A_2$. Prove that $A_1 = A_2$.
ii) Suppose E is an open set in $\mathbb{R}^n$, f maps E into $\mathbb{R}^m$, f is differentiable at $x_0 \in E$, g maps an open set containing $f(E)$ into $\mathbb{R}^k$, and g is differentiable at $f(x_0)$. Prove that the mapping F of E into $\mathbb{R}^k$ defined by $F(x) = g(f(x))$ is differentiable at x_0, and $F'(x_0) = g'(f(x_0))f'(x_0)$.
iii) State and prove Inverse function theorem.

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