DHANAMANJURI UNIVERSITY

Examination- 2024 (December)

M.Sc 1st Semester

Name of Programme : M.Sc. Mathematics
Paper Type : Theory
Paper Code : MAT-501
Paper Title : Advanced Abstract Algebra-I
Full Marks : 80
Pass Marks : 32 **Duration: 3 Hours**

The figures in the margin indicate full marks for the questions:

UNIT-I

Answer any three from the following questions: $10 \times 3 = 30$

1. Show that a normal H of a group G is maximal iff $\frac{G}{H}$ is simple. Let H and K be two distinct normal subgroups of a group G, then show that $G = HK$ and $H \cap K$ is a maximal normal subgroup of H as well as K. 5+5 = 10

2. Define composition series. Prove that any two-composition series of a finite group are equivalent. 1+9=10

3. Prove that every finite group G has a composition series. Also, find all the composition series of a cyclic group of order 6 and show they are all equivalent. 5+5=10

4. Define solvable group with an example. Prove that a group G is solvable iff $G^{(n)} = \{e\}$ for some positive integer n, where $G^{(n)}$denotes the n^{th} derived group of G. Also, show that $S_n (n \geq 5)$ is not solvable. 2+6+2=10

5. Let N be a normal subgroup of a group G such that N and $\frac{G}{N}$ are solvable, then show that G is solvable. Moreover, prove that a finite group is solvable iff its composition factors are cyclic groups of prime orders. 5+5=10

6. Prove that a group G is nilpotent iff G has a normal series $\{e\} = G_0 \subseteq G_1 \subseteq ...G_n = G$ such that $\dfrac{G_1}{G_{i-1}} \subseteq Z(\dfrac{G}{G_{i-1}}), \forall i = 1, 2, ..., n$. Show that every nilpotent group is solvable but the converse need not be true.

UNIT-II

Answer any 3 (three) from the following questions: $10 \times 3 = 30$

1. If $a \in K$ is algebraic over F, then prove that there exists a unique monic polynomial $p(x) \in F[x]$ such that $p(a) = 0$. Further, if $f(x) \in F[x]$ with $f(a) = 0$, then show that $p(x)|f(x)$. 6+4=10

2. Define minimal polynomial of any element $a \in K$, an extension field of F. Let K be an intension field of F and $a \in K$ be an algebraic of degree n. Then prove that $F(a) = \{\beta_0 + \beta_1 a + \beta_2 a^2 + ... + \beta_{n-1}a^{n-1} | \beta_i \in F, \forall i = 0, 1, 2, ..., n-1\}$. Also, show that each element of $F(a)$ is unique. 2+6+2=10

3. Let K be an extension field of F, then prove that an element $a \in K$ is algebraic if and only if $[F(a) : F]$ is finite. 4+6=10

4. Define field extension of a field F and given an example.Let K be a finite field extension of F and let L be a finite field extension of K, then prove that L is a finite extension of F and $[L : F] = [L : K][K : F]$. 2+8=10

5. Prove that every finite extension of a field F is an algebraic extension but the converse is not true, in general. 6+4=10

6. Define multiplicity m of $f(x) \in F[x]$. Proof that a non- zero polynomial $f(x)$ of degree n over a finite F can have almost n roots in any field extension of F. 2+8=10

UNIT-III

Answer any 2 (two) from the following questions: $10 \times 2 = 20$

1. Let E be a Galois extension of a field F. Let K be any subfield of E containing F. Then, show that the mapping $K \to G({}^{E}/_{F})$ set up a one-one correspondence from the set of subfields of E containing F to the subgroups of $G({}^{E}/_{F})$ such that 4+3+3=10
   i) $K = E_{G(E/K)}$
   ii) For any subgroup H of $G({}^{E}/_{F})$, $H = G(E/E_H)$.
   iii) $[E : k] = |G(E/K)|$, $[K : F] =$ index of $G(E/K)$ in $G(E/F)$.

2. Prove that a polynomial $f(x) \in F[x]$ is solvable by radicals over F if and only if its splitting field E over F has solvable Galois group $G({}^{E}/_{F})$. 5+5=10

3. i) Show that the n^{th} cyclotomic polynomial $\Phi_n(x) = \prod_{\omega}(x - \omega)$, where ω is the n^{th} root in $\mathbb{C}$, is an irreducible polynomial of degree $\phi(n)$(Euler's totient function) in $\mathbb{Z}[x]$.

   ii) Show that a polynomial $x^5 - 9x + 3$ is not solvable by radicals over $\mathbb{Q}$.

5+5=10.

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