DHANAMANJURI UNIVERSITY
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Name of Programme : B.A./B.Sc. Mathematics
Semester : 5th
Paper Type : DSE
Paper Code : EMA-001
Paper Title : Metric Space
Full Marks : 80
Pass Marks : 32 Duration: 3 Hours

The figures in the margin indicate full marks for the corresponding questions.
Answer all questions:

1. Choose the correct answer from the following and rewrite it : $1 \times 3 = 3$

a) In the usual metric space $\mathbb{R}_u$, the open sphere $S_\delta(a)$ is equal to

(i) $[a-\delta, a+\delta]$
(ii) $[a-\delta, a+\delta)$
(iii) $(a-\delta, a+\delta]$
(iv) $(a-\delta, a+\delta)$

b) Consider the following statements:
S_1: In a metric space, every convergent sequence has a unique limit.
S_2: In a metric space, every bounded sequence is convergent.
Then

(i) only S_1 is true.
(ii) only S_2 is true.
(iii) both S_1 and S_2 are true.
(iv) neither S_1 nor S_2 is true.

c) Consider the following statements:
Which of the following sets is compact in the usual metric space $\mathbb{R}_u$?

(i) $[0, 2] \cap \mathbb{Q}$
(ii) $[0, 2] \cap \mathbb{Q}^c$
(iii) $\{1, 2, 3, \ldots, 10\}$
(iv) $\mathbb{R}$

2. Write very short answer for each of the following questions: **$1 \times 6 = 6$**

a) Write the distance between the subsets $[-3, -2] \cup [2, 3]$ and $[0, 1]$ of the usual metric space $\mathbb{R}_u$.

b) Give an example to show that under continuous mappings Cauchy sequences are not necessarily preserved.

c) State Cantor's Intersection Theorem.

d) Consider the discrete metric space X_d, where X is an infinite set. Write an open cover of X containing proper subsets of X.

e) When is a metric space said to have Bolzano-Weierstrass Property?

f) When is a subset A of a metric space (X, d) said to be an ε-net, where $\varepsilon > 0$?

3. Answer the following questions: **$3 \times 5 = 15$**

a) Prove that in a metric space, the complement of any singleton set is open.

b) Show that the closure of a set in a metric space is a closed set .

c) Let (X, d) and (Y, ρ) be metric spaces and $f : X \to Y$ be a uniformly continuous function. If $\{x_n\}$ is a Cauchy sequence in X, then prove that $\{f(x_n)\}$ is a Cauchy sequence in Y.

d) Let (X, d) be a metric space and let $A, B \subset X$ be compact. Prove that $A \cup B$ is compact .

e) Show that the space $C[0, 1]$ is not compact . (where the symbol has usual meaning)

4. Answer the following questions: **$4 \times 5 = 20$**

a) Let (X, d) be a metric space. For $x, y \in X$, define

$$d^*(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$

Prove that (X, d^*) is a metric space.

b) Let (X, d) be a metric space and $A \subset X$. Prove that A is closed if and only if A contains all of its limit points.

c) Show that in a metric space, every convergent sequence is a Cauchy sequence but the converse is not necessarily true.

d) Let (X, d) and (Y, ρ) be metric spaces and $f : X \to Y$ be a function. Prove that f is continuous if and only if $f^{-1}(B) \subset f^{-1}(\overline{B})$, for every subset B of Y.

e) Let (X, d) and (Y, ρ) be metric spaces and $f : X \to Y$ be a continuous function. Prove that if X is compact, then f is uniformly continuous.

5. Answer any two of the following questions: **$6 \times 2 = 12$**

a) Let (X, d) be a metric space. Prove that

(i) Arbitrary union of open sets in X is open.

(ii) Finite intersection of open sets in X is open.

b) Let (X, d) be a metric space and $A \subset X$. Prove that

(i) A° is an open set.

(ii) A° is the largest open subset of A.

(iii) A is open $\iff A = A^\circ$.

c) Let (Y, d_Y) be a subspace of a metric space (X, d) and $A \subset Y$. Prove that

(i) $x \in Y$ is a limit point of A in Y if and only if x is a limit point of A in X.

(ii) The closure of A in Y is $\overline{A} \cap Y$, where $\overline{A}$ is the closure of A in X.

6. Answer any two of the following questions: **$6 \times 2 = 12$**

a) Show that the sequence space l^p, $1 \leq p < \infty$, is a complete metric space.

b) Let (X, d) and (Y, ρ) be metric spaces and $f : X \to Y$ be a function. Prove that f is continuous if and only if $f^{-1}(G)$ is open in X whenever G is open in Y.

c) Let (X, d) and (Y, ρ) be two metric spaces and $f : X \to Y$ be a bijection. Show that the following statements are equivalent:

(i) f is a homeomorphism.

(ii) The set $G \subset X$ is open if and only if its image $f(G) \subset Y$ is open.

(iii) The set $F \subset X$ is closed if and only if its image $f(F) \subset Y$ is closed.

7. Answer any two of the following questions: **$6\times 2 = 12$**

a) Show that a compact subset of a metric space is closed and bounded.

b) Prove that in a sequentially compact metric space, every open cover has a Lebesgue number.

c) Show that a metric space (X, d) is compact if and only if every collection of closed subsets of X having finite intersection property has non-empty intersection.

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