DHANAMANJURI UNIVERSITY

Examination- 2025 (December)

Name of Programme : B.A./B.Sc. Mathematics
Semester : 7th
Paper Type : DSE
Paper Code : EMA-005
Paper Title : Advanced Ordinary Differential Equation

Full Marks : 80
Pass Marks : 32 **Duration: 3 Hours**

The figures in the margin indicate full marks for the corresponding questions.

1. **Answer any 3 (three) of the following questions: $10 \times 3 = 30$**

   a) Define an ordinary point. Solve in power series the Legendre's equation $(1-x^2)y'' - 2xy' + p(p+1)y = 0$ where p is a constant about the point $x = 0$. **1+9=10**

   b) Define an indicial equation. Solve in series the equation $(1-x^2)y'' - xy' + 4y = 0$ near the point $x = 0$. **1+9=10**

   c) i) Prove that if $y(x)$ is a solution of the IVP: $y' = f(x, y),\ y(x_0) = y_0$ on an interval I containing x_0 if and only if it is a continuous solution of the integral equation $y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\,dt$ on I, where f is continuous in a domain D containing the point (x_0, y_0).

   ii) Using Picard's method of successive approximations, find the first three approximations of the solution of the equation $\frac{dy}{dx} = x + y^2$ where $y = 0$ when $x = 0$. **5+5=10**

d) Define Lipschitz continuity. Prove that if f is continuous and satisfy a Lipschitz condition in $R=\{(x,y):|x-x_0|\le a,|y-y_0|\le b, a>0, b>0\}$ and if in the IVP: $y'=f(x,y),\ y(x_0)=y_0$, the initial value changes by a small amount, say $|y_0-\bar{y}_0|<\delta$, then the solution also changes accordingly by a small amount, that is, $|y(x)-\bar{y}(x)|<\varepsilon$. **1+9=10**

e) State Picard's Existence Theorem. Consider the initial value problem $\frac{dy}{dx}=x^2+y^2,\ y(0)=0$ on $R=\{(x,y):0\le x\le a,|y|\le b, a>0, b>0\}$. Show that Picard's theorem guarantees the existence of a solution of the above IVP on the interval $|x|\le\frac{1}{\sqrt{2}}$.

2. Answer any 2 (two) of the following questions: $10\times 2=20$

a) i) Represent the following linear equation $u''+3v'+4u+5v=6x$ and $v''-u'+4u+v=\cos x$ in vector matrix form.

ii) Prove that if the Wronskian of two solutions of the differential equation $a_0(x)y''+a_1(x)y'+a_2(x)y=0$ where $a_0(x),\ a_1(x),\ a_2(x)$ are continuous and $a_0\ne 0\ \forall x\in(a,b)$ is linearly independent, then $W(x)=Ae^{-\int\frac{a_1(x)}{a_0(x)}dx}$. **5+5=10**

b) i) Use Wronskian to show that the functions x, x^2 and x^3 are linearly independent. Determine the differential equation with these as an independent solution.

ii) Prove that two solutions of the linear homogeneous second order differential equation $a_0(x)y''+a_1(x)y'+a_2(x)y=0$ where $a_0(x),\ a_1(x),\ a_2(x)$

are continuous and $a_0 \neq 0\ \forall x \in (a,b)$ are linearly dependent if and only if their Wronskian is identically zero.

5+5=10

c) Define Wronskian. Show that the solutions e^x, e^{2x}, e^{-2x} of $y''' - y'' - 4y' + 4y = 0$ are linearly independent and hence or otherwise solve the given equation. 1+9=10

d) Consider the system $Y' = AY + B$ where $A = \begin{pmatrix} 3 & 2 \\ 0 & 3 \end{pmatrix}$,

$B = \begin{pmatrix} e^x \\ e^{-x} \end{pmatrix}$. Show that $\varphi(x) = \begin{pmatrix} e^{3x} & 2xe^{3x} \\ 0 & e^{3x} \end{pmatrix}$ is a fundamental matrix. Compute a solution Y of the system with $Y(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

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3. Answer any 3 (three) of the following questions: $10 \times 3 = 30$

a) i) Prove that every second order linear differential equation can be put in self-adjoint form.

ii) Find the self-adjoint form of the Legendre equation $(1-x^2)y'' - 2xy' + n(n+1)y = 0$.

iii) Find the self-adjoint form of the Bessel equation $x^2y'' + xy' + (x^2 - n^2)y = 0$. 6+2+2=10

b) If $f(x)$ and $g(x)$ are real solutions of

$$\frac{d}{dx}\left[P(x)\frac{dy}{dx}\right] + Q_1(x)y = 0 \quad \text{and} \quad \frac{d}{dx}\left[P(x)\frac{dy}{dx}\right] + Q_2(x)y = 0$$

on the interval $[a,b]$ respectively such that $P(x) > 0$, $Q_2(x) > Q_1(x)$ and if x_1 and x_2 are successive zeros of $f(x)$ on $[a,b]$, prove that $g(x)$ has at least one zero at some point of the open interval $x_1 < x < x_2$. 10

c) Prove that if $f(x)$ and $g(x)$ are two linearly independent solutions of $L_2[y] \equiv \frac{d}{dx}\left[P(x)\frac{dy}{dx}\right] + Q(x)\, y = 0$ on an interval $I = [a,b]$, then between any two successive zeros of $f(x)$, there exists exactly one zero of $g(x)$. Show with the help of an example that the number of zeros of linearly independent solutions of $L_2[y] = 0$ on an interval I differ at most by one.

8+2=10

d) Find the eigenvalues and eigenfunctions of the Sturm-Liouville system **10**

$y'' + \lambda y = 0, \ 0 \le x \le \pi$

$y(0) = 0, \ y'(\pi) = 0$

e) What is a periodic Sturm-Liouville system. Define eigenvalues and eigenfunctions of a Sturm-Liouville system. Prove that the eigenfunctions of a periodic Sturm-Liouville system in $[a,b]$ are orthogonal with respect to the weight function $s(x)$ in $[a,b]$. **1+2+7=10**

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