DHANAMANJURI UNIVERSITY

Examination- 2025 (December)

Name of Programme : B.A./B.Sc. Mathematics
Semester : 7th
Paper Type : DSE
Paper Code : EMA-003
Paper Title : Field Theory and Galois Theory
Full Marks : 80
Pass Marks : 32
Duration: 3 Hours

The figures in the margin indicate full marks for the questions.

Answer any 3 (three) of the following questions: $10 \times 3 = 30$

1. a) Show that field homomorphism is either zero or 1-1. 3

   b) Let F be a field and $p(x) \in F[x]$ be an irreducible polynomial. Suppose α be a root of $p(x)$ in some extension of F. Then prove that $F(\alpha) \cong \frac{F(x)}{<(p(x)>}$. 7

2. a) Prove that every prime subfield of a field is isomorphic to $\mathbb{Q}$ or $\frac{\mathbb{Z}}{p\mathbb{Z}}$ for some prime p. 6

   b) Find the splitting field of $x^2 - 2$ over $\mathbb{Q}$. 4

3. Let $\phi : F \to F'$ be an isomorphism of fields. Let $f(x) \in F[x]$ be a polynomial, and $f'(x) \in F'(x)$ be the polynomial obtained by applying ϕ to the coefficient of $f(x)$. Then prove that $E \overset{\sigma}{\cong} E'$ and σ restricted to $F = \phi$ where E and E' are the splitting fields of $f(x)$ and $f'(x)$ respectively.

4. a) Let E be a field extension of F and $u \in E$ is algebraic over F. Let $p(x) \in F[x]$ be a polynomial of least degree such that $p(u) = 0$. Show that

   i) $p(x)$ is irreducible over F.

   ii) If $g(x) \in F[x]$ such that $g(u) = 0$, then $p(x) | g(x)$.

iii) There is exactly one monic polynomial $p(x) \in F[x]$ of least degree such that $p(u) = 0$. 7

b) Prove that every irreducible polynomial over a field of char 0 is separable. 3

5. Let $L = F(\alpha_1, \alpha_2, \ldots\ldots\ldots, \alpha_n)$ be a finite extension of F where each α_i is separable over F. Then prove that there exists $\alpha \in L$ separable over F such that $L = F(\alpha)$. Furthermore, if F is infinite, then show that α can be chosen to be of the form

$$\alpha = t_1\alpha_1 + t_2\alpha_2 + \ldots\ldots\ldots + t_n\alpha_n \text{ where } t_1 \in F.$$

Answer any 3 (three) of the following questions: $10 \times 3 = 30$

6. Prove that $K|F$ is Galois iff K is the splitting field of some separable polynomial over F. Further, if this is the case then every irreducible polynomial in $F[x]$ which has a root in K is separable and has all its roots in K.

7. Let $G = \{\sigma_1 = I, \sigma_2, \ldots, \sigma_n\}$ be a subgroup of the automorphisms of a field K and let F be its fixed field. Then, show that $[K:F] = n = |G|$.

8. a) Prove that distinct subgroups of the automorphisms of a field K has distinct fixed fields. 3

   b) Find $Gal(\mathbb{Q}\sqrt{2}, \sqrt{3})|\mathbb{Q})$. Also find all the subgroups of the Galois group $Gal(\mathbb{Q}\sqrt{2}, \sqrt{3},)/\mathbb{Q})$ and its corresponding fixed fields. 7

9. a) If L is the splitting field of a separable polynomial $f(x) \in F[x]$ of degree n, then prove that there is a 1-1 group homomorphism from $Gal(L|F) \rightarrow S_n$. 6

   b) Let $F \subset L$ be a finite extension and $\sigma \in Gal(L|F)$. Then, prove that for any non constant polynomial $h(x) \in F[x]$ with $\alpha \in L$ as a root, $\sigma(\alpha)$ is another root of $h(x)$ lying in L. Further, show that if $L = F(\alpha_1, \alpha_2, \ldots\ldots\ldots \alpha_n)$ then σ is uniquely determined by its values on $\alpha_1, \alpha_2, \ldots\ldots\ldots \alpha_n$. 4

10. Let K be a Galois extension of F and $G = Gal(K/F)$. Then prove that there is a bijection between the subfields of K containing F and subgroups of G, where any subfield L corresponds to the

Galois group $Gal(K/L)$ and any subgroup H of G corresponds to the field of elements fixed elementwise by H:

$L \rightarrow$ {automorphism in G fixing L elementwise} and

{the fixed field of H}$\leftarrow H$.

and $[K : L] = |H|$ and $[L : F] = |G : H|$, the index of H in G. Furthermore, prove the following

i) The correspondence is inclusion reversing i.e. if the fields L_1 and L_2 are associated with the subgroups H_1 and H_2, then $L_1 \subseteq L_2$ iff $H_2 \leq H_1$.

ii) K is always Galois over L with $Gal(K|L) = H$.

iii) L is Galois over F iff H is a normal subgroup of G.

Answer any 2 (two) of the following questions: $10 \times 2 = 20$

11. a) Let F be a field and n be a positive integer. Then prove that there exists a primitive n^{th} root of unity in some extension E of F iff either char F=0 or char $F \nmid$ n. 7

b) Prove that any two finite fields with p^n elements are isomorphic. 3

12. Define n^{th} Cyclotomic polynomial. Prove that n^{th} Cyclotomic Polynomial is an irreducible polynomial of degree $\phi(n)$ in $\mathbb{Z}[x]$.

13. If E_r is a radical extension of $F = E_0$ with intermediate fields $E_1, E_2, \ldots, E_{r-1}$ written in ascending order, then prove that there exists a radical extension E_s' of $F = E_0$ with intermediate fields $E_1', E_2', \ldots, E_{s-1}'$ written in ascending order such that

i) $E_r \subset E_s'$

ii) E_s' is a normal extension of F

iii) E_i' is a splitting field of a polynomial of the form $x_i^m - b_r \in E_{i-1}'[x], i=1,2,\ldots\ldots,n$

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