DHANAMANJURI UNIVERSITY

Examination- 2024 (Dec)

Four year course B.Sc./B.A. 5^{th} Semester

Name of Programme	:	**B.Sc./B.A. Mathematics**
Paper Type	:	**DSE(Theory)**
Paper Code	:	**EMA-001**
Paper Title	:	**Metric Space**

Full Marks : 80

Pass Marks : 32 **Duration: 3 Hours**

The figures in the margin indicate full marks for the questions:
Answer all the question.

1. Choose and rewrite the correct answer for each of the following: **$1 \times 3 = 3$**

a) In the discrete metric space X_d, the closed sphere $S_r[x] = \{x\}$ if

i) $0 < r \leq 1$ ii) $0 < r < 1$
iii) $1 < r$ iv) $1 \leq r$

b) The property of convergence of a sequence in a metric space (X, d)

i) Does not depend on the space X.
ii) depends only on the space X.
iii) depends only on the metric used.
iv) depends on the space X and the metric used.

c) Consider the following Statements:

S_1 : A fine set in any metric space is compact
$S2$: The usual metric space R_u is compact.

Then

i) Only S_1 is true.
ii) Only S_2 is true.
iii) both S_1 and S_2 are true.
iv) neither S_1 nor S_2 is true.

2. Write very short answer for each of the following questions : $1 \times 6 = 6$

a) Define dense subset.

b) Give an example of a Cauchy sequence that is not convergent.

c) Show that the metric spaces [0,1] and [0,3] with the usual metric are homomorphic.

d) Define cover of a metric space.

e) Is the set $\{0, 1, \frac{1}{2}, \frac{1}{3}, ...\}$ in the usual metric space $\mathbb{R}_u$ compact ? Justify your answer.

f) Define Lebesgue number.

3. Write short answer for each of the following questions: $3 \times 5 = 15$

a) Let X, d be a metric space. Prove that $|d(x, z) - d(z, y)| \leq d(x, y)$.

b) Prove that every finite set in a metric space is closed.

c) Show that a Cauchy sequence in a metric space is convergent if it has a convergent subsequence.

d) Let (X, d) be a metric space and let $A, B \subset X$ be compact. Prove that $A \cap B$ is compact.

e) Let (X, d) and $(Y,)$ be metric space and $f : X \to Y$ be a continuous function. If $A \subset X$ is compact in X, then prove that $f(A)$ is compact in Y.

4. Answer the following questions: $4 \times 5 = 20$

a) For $x = (\alpha_1, \alpha_2, ...\alpha_n)$ and $y = (\beta_1, \beta_2, ..., \beta_n)$ in $\mathbf{R}^n$,define
$d(x, y) = \{\sum_{i=1}^{n}(\alpha_1 - \beta_1)^2\}^{\frac{1}{2}}$. Prove that $(\mathbf{R}^n, d)$ is a metric space.

b) Prove that every open sphere in a metric space is an open set.

c) Let (X, d) and $(Y,)$ be metric spaces and $f : X \to Y$ be a function. Prove that f is continuous if and only if $f(A) \subset \overline{f(A)}$, for every subset A of X.

d) Let (X, d) be a metric space and $A \subset X$. Then, show that the function
$f : X \to \mathbf{R}$ given by
$f(x) = d(x, A), x \in X$ is uniformly continuous.

e) Prove that in a compact metric space every closed subset is compact.

5. Answer any two of the following questions: **$6 \times 2 = 12$**

a) Let (X, d) be a metric space and let $A, B \subset C$.Then prove that

i) $A \subset B \Rightarrow A^\circ \subset B^\circ$.

ii) $(A \cap B)^\circ = A^\circ \cap B^\circ$.

iii) $(A \cup B)^\circ = A^\circ \cup B^\circ$.

b) Let X, d be a metric space. Then prove that

i) Arbitrary intersection of closed sets in X is closed.

ii) Finite union of closed sets in X is closed.

c) Let (Y, d_Y) be a subspace of a metric space X, d and $A \subset X$. Then prove that

i) A is open in Y if and only if $\exists$ an open set G in X such that $A = G \cap Y$.

ii) A is closed in Y if and if $\exists$ a closed set F in X such that $A = F \cap Y$.

6. Answer any two of the following questions: **$6 \times 2 = 12$**

a) Show that the sequence space l^∞ is a complete metric space.

b) Show that metric space $(C[0, 1], d_1)$, where $d_1(x, y) = \int_0^1 |x(t) - y(t)|dt$, is not complete.

c) Let (X, d) be a complete metric space and let $\{F_n\}$ be a decreasing sequence of non-empty closed subsets of X such that $d(F_n) \to 0$. Then prove that the intersection $\cap_{n=1}^{\infty} F_n$ contains exactly one point.

7. Answer any two of the following questions: **$6 \times 2 = 12$**

a) Prove that a metric space is sequentially compact if and only if it has the Bolzano-Weierstrass Property.

b) Let (X, d) be a metric space. Prove that X is totally bounded if and only if every sequence in X contains a Cauchy sequence.

c) Show that a metric space (X, d) is compact if only if every collection of closed subsets of X having finite intersection property has non-empty intersection.

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