

DHANAMANJURI UNIVERSITY

DECEMBER 2025

Name of Programme : B.A./B.Sc. Mathematics
Semester : 5th
Paper Type : Core
Paper Code : CMA-315
Paper Title : Group Theory - II
Full Marks : 80
Pass Marks : 32 **Duration: 3 Hours**

The figures in the margin indicate full marks for the corresponding questions.

Answer all questions:

1. Choose the correct answer from the following and rewrite it : **1 × 3 = 3**

(a) The group of automorphisms of an infinite cyclic group is of order

(i) only one (ii) exactly two

(iii) infinite (iv) zero

(b) Let G be any group and take $S = G$. Define $*$ by $a * x = axa^{-1} \forall a, x \in G$. This action of G on itself is called

(i) right translation (ii) left translation

(iii) conjugation (iv) stabilizer.

(c) If A, B are finite cyclic groups of order m and n respectively, then $A \times B$ is cyclic if

(i) $m \leq n$ (ii) $m \geq n$

(iii) $m = n$ (iv) $(m, n) = 1$.

2. Write very short answer for each of the following questions: **1 × 6 = 6**

1. When a subgroup H of a group G is called characteristic subgroup?
2. Write the correct relationship among $A(G)$, $\text{Aut}(G)$ and $I(G)$ associated with an arbitrary group G .
3. Define Sylow's p -subgroups of a group G .
4. Define External Direct Product of two groups.
5. State the generalized Cayley Theorem.
6. When a group G is said to be simple?

3. Write short answers for each of the following:**3 × 5 = 15**

- (a) Let G be a group that acts on a set S . Let $x \in S$, then show that there exists a one-one onto map from G_x to the set of all left cosets of G_x in G .
- (b) If $cl(a)$ denotes conjugate class of a in G , then prove that $cl(a) = \{a\} \Leftrightarrow a \in Z(G)$.
- (c) Show that a homomorphism from a simple group is either trivial or one-to-one.
- (d) Suppose a group G is an IDP of its subgroups H and K . Prove that H and K have only the identity in common.
- (e) If a group G has only one p -subgroup H , then prove that H is normal in G .

4. Answer the following questions:**4 × 5 = 20**

- (a) Suppose a group G acts on two sets S and T with $*$ defined by $g*(s, t) = (gs, gt)$ is a G -action on $S \times T$. Prove that stabilizer of (s, t) is the intersection of the stabilizers of s and t .
- (b) If G' is the commutator subgroup of a group G , prove that
- G' is normal in G ,
 - G/G' is abelian.
- (c) If $o(G) = 216$, then prove that G is not simple.
- (d) Suppose X is a conjugate class of a non-trivial elements of G and let $T \in \text{Aut}(G)$. Prove that $T(X) = \{T(x) : x \in X\}$ is a conjugate class of elements of G .
- (e) Show that A_5 cannot contain a subgroup of order 30, 20 or 15.

5. Answer any two of the following questions:**6 × 2 = 12**

- (a) Let G be an infinite cyclic group. Prove that $\text{Aut}G \cong \mathbb{Z}_2$, symbols have usual meaning.
- (b) Let $f : G \rightarrow G$ be a homomorphism. Suppose f commutes with every inner automorphism of G . Show that
- $K = \{x \in G : f^2(x) = f(x)\}$ is a normal subgroup of G ,
 - G/K is abelian.
- (c) Suppose $a \in G$ has only two conjugates in G , then show that the number of distinct elements $g \in G$ such that $g^{-1}xg = y$ is $o(N(x))$.

6. Answer any two of the following questions:**6 × 2 = 12**

- (a) Let $o(G) = pq$, where p, q are distinct primes such that $p < q, p \nmid q - 1$. Show that G is cyclic.
- (b) Let G be a finite group and p is the smallest prime divisor of $o(G)$. Show that a subgroup H of index p in G is normal in G .
- (c) Prove that a normal subgroup H of G is maximal if and only if the quotient group G/H is simple.

7. Answer any two of the following questions:**6 × 2 = 12**

- (a) Let H_1, H_2, \dots, H_n be normal in G . Then G is an IDP of H_1, H_2, \dots, H_n if and only if
- (i) $G = H_1 H_2 \cdots H_n$
 - (ii) $H_i \cap H_1 H_2 \cdots H_{i-1} H_{i+1} \cdots H_n = \{e\}$ for all $i = 1, 2, \dots, n$.
- (b) Let p be a prime and m is a positive integer such that p^m divides $o(G)$. Then show that there exists a subgroup H of G such that $o(H) = p^m$.
- (c) Let H, K are the normal subgroups of a group G . Prove that $\frac{G}{H \cap K}$ is isomorphic to a subgroup of $\frac{G}{H} \times \frac{G}{K}$.
