DHANAMANJURI UNIVERSITY
DECEMBER 2025

Name of Programme	**:**	**B.A./B.Sc. Mathematics**
Semester	**:**	**5^{th}**
Paper Type	**:**	**Core**
Paper Code	**:**	**CMA-314**
Paper Title	**:**	**Ring Theory and Linear Algebra - I**

Full Marks : 80

Pass Marks : 32 **Duration: 3 Hours**

The figures in the margin indicate full marks for the corresponding questions.
Answer all questions:

1. Choose the correct answer from the following and rewrite it : **$1 \times 3 = 3$**

(i) Which of the following is a commutative ring with unity?

(A) The ring of integers

(B) The ring of 3×3 matrices over integers

(C) The ring of even integers

(D) None of the above.

(ii) In the linear space $\mathbb{R}^n$ ($\mathbb{R}$), a basis subset consists of exactly

(A) n vectors including the zero vector

(B) (n+1) non zero vectors

(C) (n-1) non zero vectors

(D) n non zero vectors

(iii) Let $T : \mathbb{R}^3 \to \mathbb{R}$ be a linear transformation defined by
$T(x, y, z) = (x, y + z, z - x)$.
The condition for $(x, y, z) \in \text{Ker}\, T$ is

(A) $x = 0, y = 0, z \neq 0$

(B) $x = 0, y \neq 0, z = 0$

(C) $x = 0, y = 0, z = 0$

(D) $x \neq 0, y = 0, z = 0$

2. Write very short answer for each of the following: **1×6 = 6**

(A) Give an example of a subring which is not an ideal.

(B) Define an integral domain.

(C) Is the union of two subspaces a subspace?

(D) Write the statement of the fundamental theorem of vector space homomorphism.

(E) When is a linear transformation $T : V(F) \to W(F)$ said to be non-singular?

(F) What is the zero vector of the vector space $\mathrm{M}_{m\times n}(\mathbb{R})$ of all $m \times n$ rectangular matrices over $\mathbb{R}$?

3. Write short answer for each of the following: **3×5 =15**

(A) Let R be a ring with unity. Show that if 1 is of additive order n then ch $R = n$ and if 1 is of additive order infinity then ch $R = 0$.

(B) Show that a non-empty subset S of a ring R is a subring of R iff $a, b \in S \Rightarrow a - b, ab \in S$.

(C) Show that in $V(\mathbb{R}) \equiv \mathbb{R}^4$, the subset $W_1 = \{(x, y, z, u) \mid x, y, z, u \in \mathbb{R},\ x + z = y + u\}$ forms a subspace of $V(\mathbb{R})$.

(D) Show that in a vector space $V(F)$ every superset of a L.D. set is L.D.

(E) Find a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ such that the range of T is spanned by the set $\{(1, 0, -1), (1, 2, 3)\}$.

4. Write short answer for each of the following: **4×5 = 20**

(A) Prove that in a ring R

(i) $a.0 = 0.a = 0\ \forall\ a \in R.$

(ii) $a.(-b) = (-a).b = -a.b\ \forall\ a, b \in R.$

(iii) $(-a).(-b) = a.b\ \forall\ a, b \in R.$

(iv) $a.(b - c) = a.b - a.c\ \forall\ a, b, c \in R.$

(B) If A and B are two ideals of a ring R then show that $A+B$ is an ideal of R containing both A and B.

(C) Show that the set $\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$ where

$$\varepsilon_1 = (1, 1, 2), \varepsilon_2 = (-3, 1, 0), \varepsilon_3 = (1, -1, 1), \varepsilon_4 = (1, 2, -3)$$

forms a linearly dependent (L.D) set of $\mathbb{R}^3(\mathbb{R})$.

(D) Let $S = \{v_1, v_2, \ldots, v_n\}$ be a linearly independent subset of a vector space $V(F)$ and let $v \in V(F)$ be such that $v \notin L(S)$. Prove that $S_1 = \{v, v_1, v_2, \ldots, v_n\}$ is also linearly independent (L.I).

(E) Find range, rank, Kernel and nullity of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ as defined by $T(x_1, x_2) = (x_1 + x_2, x_1)$.

5. Answer any two (2) of the following: 6×2 = 12

(A) Show that the centre of the ring $R, Z(R)$ is a subring of R. Further show that $Z(R)$ is a field if R is a division ring.

(B) Show that a field is an integral domain. Further show that a non-zero finite integral domain is a field.

(C) Let R be the ring of all 3×3 matrices over reals. Set

$$S = \left\{ \begin{bmatrix} x & x & x \\ x & x & x \\ x & x & x \end{bmatrix} : x \in \mathbb{R} \right\}.$$

Show that S forms a subring of R and has unity different from that of R.

6. Answer any two (2) of the following: 6×2 = 12

(A) Prove that any two different bases of a FDVS have equal number of basis vectors.

(B) If W be a subspace of FDVS $V(F)$, prove that $\dim \frac{V}{W} = \dim V - \dim W$.

(C) If W_1 and W_2 are two subspaces of a FDVS $V(F)$, prove that $\dim(W_1 \oplus W_2) = \dim W_1 + \dim W_2$, provided $W_1 \cap W_2 = \{0\}$.

7. Answer any two (2) of the following: 6×2 = 12

(A) Let $T : V \to W$ be a linear transformation from a vector space $V(F)$ into another vector space $W(F)$. Prove that $\dim V(F) = \text{Rank}\, T + \text{Nullity}\, T$.

(B) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$. The matrix of T in standard ordered basis $\{e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)\}$ is $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

Find bases for Range T and Ker T.

(C) Let $T : \mathbb{C}^2 \to \mathbb{C}^2$ be defined by $T(x_1, x_2) = (x_1, 0)$
Let $\beta = \{(1,0),(0,1)\}$ and $\beta' = \{(1,1),(-1,2)\}$.
Find a non-singular matrix P such that $[T]_{\beta'} = P^{-1}[T]_\beta P$.

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