

# DHANAMANJURI UNIVERSITY

Examination- 2025 (December)

Name of Programme : B.A/B.Sc. Mathematics

Semester : 7<sup>th</sup>

Paper Type : Core (Theory)

Paper Code : CMA-419

Paper Title : Complex Analysis

Full Marks : 80

Pass Marks : 32

Duration: 3 Hours

*The figures in the margin indicate full marks for the questions*

1. Answer any 3 (three) of the following questions:  $10 \times 3 = 30$

- Derive the polar form of Cauchy-Riemann equations.
- If the imaginary part of an analytic function  $f(z)$  is a given harmonic function  $v(x, y)$ , then show that

$$f(z) = 2iv\left(\frac{z}{2}, \frac{z}{2i}\right) - iv(0, 0) + c. \text{ Hence or otherwise find}$$

$$f(z), \text{ where } v(x, y) = e^{-x}(y \sin y + x \cos y).$$

- For the function  $f(z)$ , defined

$$f(z) = \begin{cases} \frac{(1+i)x^3 - (1-i)y^3}{x^2 + y^2}, & \text{for } z \neq 0, \\ 0, & \text{for } z = 0 \end{cases}$$

show that the Cauchy-Riemann equations are satisfied at  $z = 0$  but the function is not differentiable at  $z = 0$ .

- Define four basic transformations. Show that every bilinear transformation maps circles or straight lines onto circles or straight lines.

- e) Let  $w_1, w_2, w_3, w_4$  be the images of the four distinct points  $z_1, z_2, z_3, z_4$  in  $Z$ -plane under a bilinear transformation  $w = T(z) = \frac{az + b}{cz + d}$ , ( $ad - bc \neq 0$ ). Then prove that the cross-ratio of  $w_1, w_2, w_3, w_4$  is equal to the cross-ratio of  $z_1, z_2, z_3, z_4$ . Find the bilinear transformation which maps the points  $z_1 = 2$ ,  $z_2 = i$  and  $z_3 = -2$  into the points  $w_1 = 1$ ,  $w_2 = i$  and  $w_3 = -1$ .

2. Answer any 3 (three) of the following questions:  $10 \times 3 = 30$

- a) Define a contour. Evaluate  $I = \int_{\gamma} z^2 dz$  where
- $\gamma$  is a circle centered at the origin,
  - $\gamma$  is the union of the horizontal segment from 0 to 1, the vertical segment from 1 to  $1 + 2i$  and declined line from  $1 + 2i$  to 0.
  - $\gamma$  is the line segment from  $\alpha$  to  $\beta$ .
  - $\gamma$  is the contour parameterized by  $\gamma(t) : z(t) = t^2 + it$ , ( $0 \leq t \leq 1$ ).
- b) State and prove Cauchy's theorem.
- c) Obtain the Taylor series expansion of the functions (i)  $\sin z$   
(ii)  $\sinh z$ , (iii)  $e^z$ , (iv)  $ze^z$  and (v)  $\frac{1}{1-z}$ .
- d) Let  $f(z)$  be analytic in a simply connected domain containing the simple closed contour  $C$ . Prove that  $f(z)$  has derivatives for all orders at each point  $z_0$  inside  $C$ , with  $f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz$ .
- e) State and prove Maximum Modulus Theorem.

3. Answer any 2 (two) of the following questions:  $10 \times 2 = 20$

a) State the Laurent's Theorem. Expand  $f(z) = \frac{1}{z^2 - 3z + 2}$  in

a Laurent's series valid for the regions (i)  $|z| < 1$ ,

(ii)  $1 < |z| < 2$ , (iii)  $|z| > 2$ . In which regions the function  $f(z)$  has no principal parts and no analytic parts.

b) Define isolated and non-isolated singularity. State and prove Riemann's Theorem on removable singularities.

c) Suppose  $f(z)$  is analytic inside and on a simple closed contour  $C$  except for isolated singularities at  $z_1, z_2, z_3, \dots, z_n$  inside  $C$ . Then show that

$$\int_C f(z) dz = 2\pi i \sum_{k=1}^n \text{Res}[f(z) : z_k]$$

Hence show that  $\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \pi$ .

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