DHANAMANJURI UNIVERSITY

Examination- 2026 (June)

Name of Programme	**:**	**B.A./ B.Sc. Mathematics**
Semester	**:**	**6th**
Paper Type	**:**	**Core**
Paper Code	**:**	**CMA-316**
Paper Title	**:**	**Ring theory and Linear Algebra-II**

Full Marks : 80

Pass Marks : 32 **Duration: 3 Hours**

The figures in the margin indicate full marks for the questions.

Answer all the questions:

1. Choose the correct one and rewrite the following: $1 \times 3 = 3$

a) The Polynomial $f(x) = 2x^2 - 2 \in Q[x]$ is

i) not primitive but reducible

ii) primitive as well as reducible.

iii) not primitive as well as irreducible.

iv) primitive as well as irreducible.

b) Let $V(F)$ be a finite dimensional vector space over the field F of scalars and $W(F)$ be a subspace of $V(F)$, then $A(A(W))$ is isomorphic to

i) $\hat{W}$ ii) $A(W)$

iii) W iv) $\widehat{A(W)}$

c) Let T be a linear operator on an inner product space $V(F)$. Then T is a normal operator defined on $V(F)$ if

i) $TT^* = I$ ii) $T = T^*$

iii) $T = T^{**}$ iv) $TT^* = T^*T$

2. Write very short answer for each of the following questions: $1 \times 6 = 6$

a) Define prime element in a commutative ring with unity.

b) Let $T = \mathbb{R}^2 \to \mathbb{R}^2$ be a linear operator defined by $T(x,y) = (x+y, x+y)$. Find eigen value of T.

c) State Bessel's inequality.

d) Define minimal polynomial for a linear operator T.

e) Show that $\|\alpha v\| = |\alpha|\|v\|$ for all $\alpha \in F$ and $v \in V$.

f) Define normal operator on an inner product space.

3. Write short answer for each of the following: $3 \times 5 = 15$

a) Prove that similar matrices have same minimal polynomial.

b) Let W_1, W_2 be subspaces of finite dimensional vector space V. Determine $A(W_1 + W_2)$.

c) Prove that $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is not diagonalizable.

d) If T is a linear operator on a finite dimensional vector space V over F and T is right invertible, then show that T is invertible.

e) Show that $\frac{\hat{V}}{A(W)} \cong \hat{W}$.

4. Write short answer for each of the following: $4 \times 5 = 20$

a) Let A be an $n \times n$ matrix over F. Prove that Row rank of $A = Column\, rank\, of\, A$.

b) Let T be a linear operator on an $n-$ dimensional vector space V. Show that the characteristics and minimal polynomials of T have the same roots.

c) Let S be an orthogonal set of non-zero vectors in an inner product space V. Show that S is a linearly independent set.

d) Let V be an inner product space V. Show that $|(u,v)| \leq \|u\|\|v\|$ for all $u, v \in V$.

e) Obtain an orthonormal basis with respect to the standard inner for the subspace of $\mathbb{R}^3$ generated by $(1,0,3)$ and $(2,1,1)$.

5. Answer any two of the following questions: $6 \times 2 = 12$

a) i) Show that a Euclidean domain possesses unity.

ii) In a principal ideal domain, show that evey non-zero prime ideal is maximal.

b) For any commutative ring R with unity, show that $\frac{R[x]}{<x>} \cong R$.

c) Prove that in a PID, an element is prime if and only if it is irreducible.

6. Answer any two of the following questions: $6 \times 2 = 12$

a) i) Let V be a finite dimensional vector space. Suppose $V = W_1 \oplus W_2$ where W_1, W_2 are subspaces of V. Show that $\hat{V} = A(W_1) \oplus A(W_2)$.

ii) Let A be $m \times n$ matrix with real entries. Prove that $A = 0 \Leftrightarrow$ trace $(A^t A) = 0$.

b) Let A be an $n \times n$ square matrix. Then A is diagonalizable if and only if A has n linearly independent eigen values.

c) Let T be a linear operator on a finite dimensional vector space $V(F)$. Then T is diagonalizable if and only if $\dim V = \dim W_1 + \cdots + \dim W_k$.

7. Answer any two of the following questions: $6 \times 2 = 12$

a) State and prove Gram-Schmidt orthogonalization process in an inner product space $V(F)$.

b) i) Let S and T be linear operators on a finite dimensional inner product space $V(K)$ and $\alpha \in K$. Then show that $(T + S)^* = T^* + S^*$.

ii) If T and S are self-adjoint operators on an inner product space V, then prove that TS is self adjoint $\Leftrightarrow TS = ST$.

c) Suppose T is a linear operator on a inner product space $V(F)$. Then show that the adjoint T^* of T exists such that $TT^* = T^*T = I$ if and only if T is unitary.

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