DHANAMANJURI UNIVERSITY

Examination- 2025 (June)

Four-year course B.A/B.Sc. 6th Semester (NEP)

Name of Programme	:	**B.A/B.Sc. Mathematics (Honours)**
Paper Type	:	**CORE (Theory)**
Paper Code	:	**CMA-316**
Paper Title	:	**Ring Theory and Linear Algebra II**
Full Marks	:	**80**
Pass Marks	:	**32**

Duration: 3 Hours

The figures in the margin indicate full marks for the questions.
Answer all the questions:

1. Choose and rewrite the correct answer for each of the following: **1×3=3**

a) The polynomial $f(x) = x^2 - 2 \in \mathbb{Z}[x]$ is

i) Primitive as well as reducible

ii) not primitive but reducible

iii) Primitive as well as reducible

iv) not primitive as well as irreducible.

b) Let R be an integral domain. Let $f(x), g(x) \in R[x]$ be such that $\deg(f(x)) = m$, $\deg(g(x)) = n$. Then $\deg(f(x).g(x))$ is

i) less than $(m + n)$

ii) less than $min\ (m + n)$

iii less than $max\ (m + n)$

iv) equal to $(m + n)$.

c) A linear transformation T:V→W is non-singular if

i) $Ker\ T = \{0\}$

ii) $Range\ T = \{0\}$

iii) $Rank\ T + Nullity\ T = dim\ V$

iv) $\dim V = \dim W$.

2. Write very short answer for each of the following questions: **1×6=6**

a) Let a and b be two non-zero elements in a Euclidean domain R. Write the condition for which a, b to be relatively prime.

b) State Eisentein's Criterion of irreducibility.

c) When is a square matrix of order $n \times n$ said to be diagonalizable?

d) When is a non-zero polynomial R[x] said to be primitive?

e) Show that $\|\alpha v\| = |\alpha|\|v\|$ for all $\alpha \in F, v \in V$.

f) Show that the set $S = \{(0,1,0), (0,0,1), (2,3,4)\}$ is linearly independent in the vector space $\mathbb{R}^3(\mathbb{R})$.

3. Answer the following questions: **3×5=15**

a) Prove that similar matrices have same characteristic polynomials.

b) Give an example to show that AB is diagonalizable and BA is not diagonalizable, where A and B are $n \times n$ matrices over F.

c) Let $c_1, c_2, \dots, c_k$ be distinct eigen values and $v_1, v_2, \dots, v_k$ be the corresponding eigen vectors of a linear operator T. Show that $v_1, v_2, \dots, v_k$ are linearly independent.

d) Let T be the linear operator on R^3 defined by

$$T(x_1, x_2, x_3) = (3x_1 + x_3, -2x_1 + x_2, -x_1 + 2x_2 + 4x_3).$$

Show that T is invertible.

e) Let $R[x]$ be the ring of polynomials over R. Prove that R is commutative if and only if $R[x]$ is commutative.

4. Answer the following questions: **4×5=20**

a) Let T be a linear operator on a finite dimensional vector space $V(F)$. Prove that $c \in F$ is an eigen value of T if and only if $T - cI$ is singular.

b) If T be a linear operator on an n-dimensional vector space V and suppose that T has n distinct characteristic values. Show that T is diagonalizable.

c) If V is a finite dimensional inner product space and W is a subspace of V, prove that $V = W \oplus W^{\perp}$.

d) Let V be a finite dimensional vector space and W, a subspace of V. Prove that

$$\dim A(W) = \dim V - \dim W.$$

e) Let S be an orthogonal set of non zero vectors in an inner product space V. Show that S is a linearly independent set.

5. Answer any two of the following questions: **6×2=12**

a) Let R be a commutative ring with unity such that $R[x]$ is a PID. Show that R is a field.

b) Show that any two non-zero elements a, b in a Euclidean domain R have a g.c.d and it is possible to write $d = \lambda a + \mu b$ for some $\lambda, \mu \in R$.

c) Prove that an element in a UFD is prime if and only if it is irreducible.

6. Answer any two of the following questions: **6×2=12**

a) Prove that an element in a PID is prime if and only if it is irreducible.

b) Let u and v be eigen vectors of T corresponding to distinct eigen values of a linear operator T on V. Show that $u + v$ cannot be an eigen vector of T.

c) Construct a diagonalizable 3×3 matix A whose eigen values are $-2, -2, 6$ and corresponding eigen vectors are $\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$.

7. Answer any two of the following questions: **6×2=12**

a) Let $V(F)$ be an inner product space. Show that

(i) $\|x+y\| \leq \|x\| + \|y\|$, for all $x, y \in V(F)$,

(ii) $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$.

b) If $\{w_1, w_2, \dots, w_m\}$ is an orthonormal set in V, then prove that

$$\sum_{i=1}^{m} (w_i, v)^2 \leq \|v\|^2, \text{ for all } v \in V$$

c) Let V be the space of all real valued continuous functions. Define $T: V \to V$ by

$$(Tf)(x) = \int_0^x f(t)dt.$$

Show that T has no eigen values.

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