DHANAMANJURI UNIVERSITY

Examination- 2024 (Dec)

Four-year course B.Sc./B.A. 5^{th} Semester

Name of Programme	**:**	**B.Sc./B.A. Mathematics**
Paper Type	**:**	**CORE XIV (Theory)**
Paper Code	**:**	**CMA-315**
Paper Title	**:**	**Group Theory-II**

Full Marks : 80

Pass Marks : 32 **Duration: 3 Hours**

The figures in the margin indicate full marks for the questions.

1. Choose and rewrite the correct answer: $1 \times 3 = 3$

a) Let G be a group and $f : G \to G$ be a mapping defined by $f(x) = x^{-1}\ \forall x \in G$. Then f is an automorphism if and only if

i) G is commutative

ii) G is non-commutative

iii) G is finite cyclic group

iv) $G \neq \{e\}$, e being the identity element of G

b) If a group G acts on a set S, then the stabilizer of x in G defined by $G_x = \{a \in G \mid a * x = x\}$ is

i) a normal subgroup of G

ii) a Cyclic subgroup of G

iii) a subgroup of G

iv) normalizer in G

c) Number of Sylow 2-subgroups of S_3 is

i) 1 ii)3

iii) 0 iv) 2

2. Write very short answers for each of the following: $1 \times 6 = 6$

a) When a homomorphism f on grouos is an automorphism?

b) Define characteristic subgroup of a group G.

c) When a group G is said to act on a non-empty set A

d) State the fundamental theorem of finite abelian group.

e) When a group G is said to be Simple?

f) Define internal direct product (IDP) of two subgroups.

3. Write short answers of the following: $3 \times 5 = 15$

a) Show that if $o(\text{Aut}(G)) > 1$, then $O(G) > 2$.

b) If a group G has only one P-Sylow subgroup H, then show that $H$is normal subgroup of G.

c) Show that a group of order 4 is either cyclic or is an IDP of two cyclic groups of order 2 each.

d) Let G be a group and G' be the commutator subgroup in G, show that G' is normal in G.

e) Let G be any group and S be any non-empty set. Take $S = G$. Define $*$ such that $a * x = ax, \forall a, x \in G$. Is $*$ a group action?

4. Answer the following questions: $4 \times 5 = 20$

a) If H is the only Sylow P-subgroup of a group G then prove that H is normal in G and also conversely.

b) Show that a homomorphism from a simple group is either trivial or one-to-one.

c) Suppose $a \in G$ has only two conjugates in G, then show that $N(a)$ is normal subgroup of G.

d) Show that $I(G)$, the group of all inner automorphisms is a normal subgroup of all automorphism of G i.e. $\text{Aut}(G)$.

e) Let H_1 and H_2 be normal in G. Then G is an IDP of H_1 and H_2 if $H_1 \cap H_2 = \{e\}$.

5. Answer any two of the following questions: 6 × 2 = 12

a) Show that a group G of order P^2, P being a prime, is either cyclic or isomorphic to the direct product of two cyclic groups, each of order P. .

b) If H and K are two normal subgroups of G such that $H \subseteq K$, prove that

$$\frac{G}{K} \cong \frac{G/H}{K/H}$$

c) Let G be a finite group, $a \in G$ then prove that

$$O(cl(a)) = \frac{O(G)}{O(N(a))}$$

where $cl(a)$ is the conjugate class of a.

6. Answer any *two* of the following questions: 6 × 2 = 12

a) Let G be a finite abelian group. Show that G is isomorphic to the direct product of its Sylow subgroups.

b) Let G be a finite group and P is the smallest prime divisor of $O(G)$. Show that a subgroup H of index P in G is normal in G.

c) Let G be a group and suppose G is the IDP of $H_1, H_2, \ldots, H_n$. Let T be the EDP of $H_1, H_2, \ldots, H_n$. Show that $G \cong T$.

7. Answer any *two* of the following questions: 6 × 2 = 12

a) Prove that the number of Sylow P-subgroups of a group G is of the form $1 + kP$ where k is a positive integer and $1 + kP$ divides $O(G)$.

b) Suppose a group G acts on two sets S and T. Show that $*$ defined by $g * (s, t) = (gs, gt)$ is a G-action on $S \times T$ and further prove that stabilizer of (s, t) is the intersection of the stabilizers of s and t.

c) If G is a finite group and H is a proper normal subgroup of largest order, prove that $\frac{G}{H}$ is simple.

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