DHANAMANJURI UNIVERSITY

Examination- 2025 (June)

Four year course B.A./B.Sc. 4^{th} Semester

Name of Programme	**:**	**B.A./B.Sc. Mathematics**
Paper Type	**:**	**Core-X(Theory)**
Paper Code	**:**	**CMA-210**
Paper Title	**:**	**Riemann Integration and Series of Function**

Full Marks : 80

Pass Marks : 32 **Duration: 3 Hours**

The figures in the margin indicate full marks for the questions:

Answer all the question.

1. Choose and rewrite the correct answer for each of the following: **$1 \times 3 = 3$**

a) The value of $\int_{-3}^{3} |x| dx$

i) 0

ii) 3

iii) 6

iv) 9

b) The improper integral $\displaystyle\int_0^1 x^{m-1}(1-x)^{n-1}dx$ converges if and only if

i) m>0,n>0.

ii) m>0,n<0.

iii) m<0,n>0.

iv) m<0,n<0.

c) The radius of convergence of power series
$\sum \frac{n+1}{(n+2)(n+3)} x^n$ is

i) 1

ii) 2

iii) 3

iv) 4

2. Write very short answer for each of the following questions : **$1 \times 6 = 6$**

a) What is a refinement of a partition?

b) State Darboux's theorem of Upper Riemann Sum.

c) Examine the convergence of improper integral $\int_0^1 \frac{1}{\sqrt{1-x^2}} dx$.

d) State Abel's test for convergence of improper integral.

e) Define point-wise convergence of sequence of functions.

f) State Cauchy's Criterion for uniform convergence.

3. Write short answer for each of the following: **$3 \times 5 = 15$**

a) Let $\int$ be a bounded function defined on $[a, b]$ and P is any partition of $[a, b]$, then prove that $L(P, f) \leq U(P, f)$.

b) If a function is monotonic on $[a, b]$,then prove that it is integrable on $[a, b]$.

c) Test the convergence of improper integral $\int_0^\infty \frac{x \tan^{-1} x}{(1+x^4)^{\frac{1}{3}}} dx$.

d) Show that $\int_{-\infty}^0 xe^{-x} dx$ is convergent.

e) Show that the sequence of functions $\{f_n\}$, where $f_n(x) = \dfrac{x}{1+nx}, \forall x \in \mathbb{R}$ is uniformly convergent in any interval $[0, b], b > 0$.

4. Write short answer for each of the following : 4 × 5 = 20

a) When is a bounded function f defined on $[a, b]$ said to be Riemann integrable on $[a, b]$? Show that the function $\int$ defined by $f(x) = \begin{cases} 3, & \text{if } x \text{ is rational} \\ -5, & \text{if } x \text{ is irrational} \end{cases}$ is bounded but not integrable on $[4, 7]$.

b) State and prove fundamental theorem of integral calculus.

c) Show that the improper integral $\displaystyle\int_a^\infty \frac{C}{x^p} dx, a > 0$, where C is a constant, converges if and only if $p > 1$.

d) State Weierstrass M- Test for uniform convergence. Show that $\displaystyle\sum \frac{1}{n^3 + n^4x^2}, x \in \mathbb{R}$ is uniformly convergent.

e) Find the radius of convergence of the power series $\displaystyle\sum \frac{(n!)}{2n!} x^{2n}$.

5. Answer any two of the following questions: 6 × 2 = 12

a) Prove that the oscillation of a bounded function f on an interval $[a, b]$ is the supremum of the set $\{|f(x_1) - f(x_2)| : x_1, x_2 \in [a, b]\}$ of numbers.

b) If f is bounded and integrable function on $[a, b]$, then prove that $|f|$ is bounded and integrable on $[a, b]$. Moreover prove that $|\int_a^b f(x)dx| \leq \int_a^b |f(x)|dx$.

c) A bounded function f, has a finite number of points of discontinuity on $[a.b]$. Prove that $f \in R[a, b]$.

6. Answer any two of the following questions: 6 × 2 = 12

a) Test the convergence of the improper integral $\int_0^\infty x^{m-1}e^{-x}dx$.

b) Show that $\displaystyle\int_3^5 \frac{x^2 dx}{\sqrt{(x-3)(5-x)}}$ is convergent with value $\dfrac{33\pi}{2}$.

c) State and prove Frullani's theorem for improper integral.

7. Answer any two of the following questions: $6 \times 2 = 12$

a) If a sequence $\{f_n\}$ converges uniformly to f on $[a, b]$ and each function f_n is integrable, then prove that f is integrable on $[a, b]$ and the sequence $\left\{\int_a^x f_n dt\right\}$ converges uniformly to $\int_a^x f dt$i.e., $\int_a^x f dt = \lim\limits_{n\to\infty} \int_a^x dt, \forall x \in [a, b]$.

b) If a series $\sum f_n$ converges uniformly to f in $[a, b]$ and x_0 is a point in $[a, b]$ such that $\lim\limits_{x\to x_0} f_n(x) = a_n \ (n = 1, 2, 3, ...)$, then prove that

(i) $\sum a_n$ converges, and

(ii) $\lim\limits_{x\to x_0} f(x) = \sum a_n$.

c) State and prove Cauchy- Hadamard theorem for power series.

*****