DHANAMANJURI UNIVERSITY
DECEMBER 2025

Name of Programme : **B.A./B.Sc. Mathematics**
Semester : **3rd**
Paper Type : **Core**
Paper Code : **CMA-209**
Paper Title : **Group Theory -I**
Full Marks : **80**
Pass Marks : **32** **Duration: 3 Hours**

The figures in the margin indicate full marks for the questions.

Answer all questions.

1. **Choose and rewrite the correct answer for each of the following:** $1 \times 3 = 3$

a) Which of the following does not form a group?
(i) Set of integers together with usual addition.
(ii) Set of rational numbers with respect to usual multiplication.
(iii) Set of even integers with respect to usual addition.
(iv) Set of real numbers with respect to usual addition.

b) If G is a finite cyclic group of order n then the number of distinct subgroups of G is
(i) the number of distinct divisors of n
(ii) the number of distinct multiples of n
(iii) zero
(iv) infinity

c) The set A_n of all even permutations of symmetric group S_n $(n \geq 2)$ is a normal subgroup of S_n such that
(i) $o(A_n) = o(S_n)$
(ii) $o(A_n) = \frac{o(S_n)}{2}$
(iii) $o(A_n) = 2\, o(S_n)$
(iv) $o(A_n) = \frac{o(S_n)}{3}$

2. **Write very short answer for each of the following:** $1 \times 6 = 6$
   a) Show that inverse of each element a in a group G is unique.
   b) Define normal subgroup of a group.
   c) Let $G = \{-1, 1, -i, i, -j, j, -k, k\}$ be the Quaternion group. What are the generators of G?
   d) Define an isomorphism.
   e) If $f: G \to G'$ is a homomorphism, then show that $f(e) = e'$, where identities $e \in G, e' \in G'$.
   f) Find the kernel of a homomorphism $f: (\mathbb{R}^*, .) \to (\mathbb{R}^+, .)$, such that $f(x) = x^2$,
   where $\mathbb{R}^* = \mathbb{R} - \{0\}$, $\mathbb{R}^+$ is the set of positive real numbers.

3. **Write short answers for each of the following:** $3 \times 5 = 15$
   a) Show that a finite semi-group in which cancellation laws hold is a group.
   b) Let $S = \mathbb{R} - \{-1\}$, $\mathbb{R}$ is the set of real numbers. Define a binary composition $*$ on S by $a * b = a + b + ab \quad \forall a, b \in S$. Then show that $(S, *)$ is a group.
   c) Show that intersection of two subgroups of a group G is again a subgroup of G.
   d) If $a \in G$ be a finite group of order n and also $a^m = e$, then show that $n|m$.
   e) If G is a finite group and H is a subgroup of G, then prove that $o(H)$ divides $o(G)$.

4. **Write answer for each of the following:** $4 \times 5 = 20$
   a) Prove that a non-empty subset H of a group G is a subgroup of G if and only if $a, b \in H \Rightarrow ab^{-1} \in H$.
   b) Let H be a subgroup of a group G and $N(H) = \{a \in G \mid aHa^{-1} = H\}$. Prove that $N(H)$ is a subgroup of G which contains H.
   c) Prove that a subgroup H of a group G is normal in G if and only if $g^{-1}hg \in H$ for
   all $h \in H, g \in G$.

d) If $f: G \to G'$ be an onto homomorphism with $K = Ker\, f$, then prove that $\frac{G}{K} \cong G'$.

e) Show that any infinite cyclic group is isomorphic to $(\mathbb{Z}, +)$, the group of integers.

5. Answer any 2(two) questions: $6 \times 2 = 12$

a) Prove that a non-empty set G together with a binary operation $'.'$ is a group iff

(i) $a(bc) = (ab)c,\ \forall\, a, b, c \in G$

(ii) For any $a, b \in G$, the equations $ax = b$ and $ya = b$ have unique solutions. 2+4=6

b) Let p be a fixed prime and let $\mathbb{Z}_p^* = \{1, 2, \ldots, p-1\}$. Then, prove that $\mathbb{Z}_p^*$ forms a group with respect to the composition multiplication modulo p '$\odot_p$'.

c) Define General Linear Group of order 2×2. Show that the set of all matrices

$$G = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R}, ad - bc \neq 0 \right\}$$

forms a non-abelian group under matrix multiplication. 1+5=6

6. Answer any 2 (two) questions: $6 \times 2 = 12$

a) Let G be a group and $N(a) = \{x \in G \mid xa = ax\ \forall\, a \in G\}$ be a normalizer of a in G. Show that $N(x^{-1}ax) = x^{-1}N(a)x$ for all $a, x \in G$.

b) Let G be a finite group whose order is not divisible by 3. Suppose $(ab)^3 = a^3b^3$ for all $a, b \in G$, then show that G is abelian.

c) Define product of two subgroups H and K of a group G. If H and K are two subgroups of a group G, then prove that $o(HK) = \frac{o(H).o(K)}{o(H \cap K)}$. 1+5=6

7. **Answer any 2(two) questions:** $6 \times 2 = 12$

a) Let G be a finite group and suppose p is a prime such that $p|o(G)$, then prove that there exists $x \in G$ such that $o(x) = p$.

b) Let H and K be two subgroups of group G, where H is normal in G, then prove that

$$\frac{HK}{H} \cong \frac{K}{H \cap K}$$

c) Prove that every group G is isomorphic to a permutation group.

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