DHANAMANJURI UNIVERSITY

Examination- 2024 (Dec)

Four-year course B.Sc./B.A. 3rd Semester

Name of Programme	:	**B.Sc./B.A. Mathematics**
Paper Type	:	**Theory**
Paper Code	:	**CMA-209**
Paper Title	:	**Group Theory-I**

Full Marks : 80

Pass Marks : 32 **Duration: 3 Hours**

The figures in the margin indicate full marks for the questions:

Answer all the question.

1. Answer the following questions: $1 \times 3 = 3$

a) The order of the group U_{15} is

i) 15 ii) 9

iii) 10 iv) 8

b) Number of generators of an infinite cyclic group is

i) exactly two ii) only one

iii) infinite iv) zero

c) The inverse of the permutation (1234) is

i) (1234) ii) (4321)

iii) (3214) iv) (1324)

2. Answer the following questions: $1 \times 6 = 6$

a) For any a, x in a group G, show that $(x^{-1}ax)^3 = x^{-1}a^3x$.

b) Define a cyclic group.

c) Let $G = \{-1, 1, -i, i, -j, j, -k, k\}$ be the Quaternion group. Find the normalizer of i in G.

d) Define even permutation of a finite set.

e) If $f: G \to G'$ is a homomorphism then show that $f(x^{-1}) = (f(x))^{-1}$.

f) Find the kernel of $f: Z_6 \to Z_6$ where $f(x) = 2x$.

3. Answer any five of the following questions: **$3 \times 5 = 15$**

a) If G is a group in which $(ab)^i = a^i b^i$ for three consecutive integers i and any a, b in G, then show that G is abelian.

b) Let $G = \{(a, b) \mid a, b \in \mathbb{Q}, a \neq 0\}$. Define a binary composition $*$ on G by $(a, b) * (c, d) = (ac, ad + b)$ $\forall a, b, c, d \in \mathbb{Q}$, $a \neq 0, c \neq 0$. Then show that $(G, *)$ is a non-abelian group.

c) Prove that a non-empty subset H of a group G is a subgroup of G if $a, b \in H \Rightarrow ab^{-1} \in H$.

d) Show that a subgroup of a cyclic group is cyclic.

e) Find the generators of Z_8 under addition modulo 8.

f) Prove that every quotient group of a cyclic group is cyclic.

g) Show that any finite cyclic group of order n is isomorphic to Z_n the group of integers addition modulo n.

4. Answer any five of the following questions: **$4 \times 5 = 20$**

a) Define centre of the group G. Also, show that a centre of a group G is a subgroup of G.

b) Prove that union of two subgroups is a subgroup if one of them is contained in the other.

c) State and prove Lagrange's theorem.

d) Show that homomorphic image of

i) an abelian group is abelian

ii) a cyclic group is cyclic

e) Prove that order of any permutation f in S_n is equal to the $l.c.m$ of the orders of the disjoint cycles of f. What is the order of the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 6 & 5 & 1 & 3 \end{pmatrix}$

f) Let $f: G \to G'$ be a homomorphism, then show that the Kernel of f, $Ker\ f$ is a normal subgroup of G.

g) If $f: G \to G'$ be an onto homomorphism with $K = Ker\ f$, then prove that $\frac{G}{K} \cong G'$.

5. Answer any two of the following questions: $6 \times 2 = 12$

a) Let G be a group. Then prove the following results

i) Identity element in G is unique.

ii) $(ab)^{-1} = b^{-1}a^{-1}$ for all $a, b \in G$, where a^{-1} denotes inverse of a.

iii) The equations $ax = b$ and $ya = b$ have unique solutions for x and y in G. 1+2+3=6

b) Write the set of all the symmetries of an equilateral triangle. Then show that this set of symmetries D_3 forms a non-abelian group under a binary composition $*$ defined by $(ab)* = a(b*), \forall\, a, b \in D_3$ where $a*$ means the effect of a on the triangle. 2+4=6

c) Define Special Linear Group of order 2×2. Show that the set of all matrices $G = \left\{ M_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \theta \in \mathbb{R} \right\}$ forms a group under matrix multiplication. 1+5=6

6. Answer any two of the following questions: **$6 \times 2 = 12$**

a) Define centralizer $C(H)$ of H in group G. Show that $C(H)$ is a subgroup of G. Also, find the $C(H)$, if $H = \{-1, 1, -i, i\}$ is a subgroup of the Quaternion group $G = \{-1, 1, -i, i, -j, j, -k, k\}$. 1+3+2=6

b) Let H and K be two subgroups of a group G and define $HK = \{hk: h \in H, k \in K\}$. Show that HK is a subgroup of G if $HK = KH$. 6

c) Show that order of a cyclic group is equal to the order of its generator. 6

7. Answer any two of the following questions: **$6 \times 2 = 12$**

a) Prove that a subgroup H of a group G is normal subgroup of G if product of two right cosets of H in G is again a right coset of H in G. 3+3=6

b) Let H be a subgroup of a group G, then prove that

i) $Ha = H \Leftrightarrow a \in H$

ii) $Ha = Hb \Leftrightarrow ab^{-1} \in H$

iii) There is always a bijective mapping between any two right cosets of H in G. 2+2+2=6

c) If H and K are two normal subgroups of group G such that $H \subseteq K$, then prove that $\frac{G}{K} \cong \frac{G/H}{K/H}$. 6

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