

DHANAMANJURI UNIVERSITY

DECEMBER 2025

Name of Programme : B.A./B.Sc. Mathematics
 Semester : 3rd
 Paper Type : Core
 Paper Code : CMA-207
 Paper Title : Theory of Real Functions
 Full Marks : 80
 Pass Marks : 32

Duration: 3 Hours

The figures in the margin indicate full marks for the questions.

1. Choose and rewrite the correct answer for each of the following questions: 1×3=3

(a) The value of $\lim_{x \rightarrow 0} \left(\frac{e^{1/x}}{e^{1/x} + 1} \right)$ is

- (i) 0 (ii) 1 (iii) -1 (iv) does not exist

(b) The function $f(x) = |2x - 6|$ is derivable on

- (i) [1, 10] (ii) [2, 10] (iii) [3, 10] (iv) [4, 10]

(c) The Taylor's series expansion of $\sin x$ is

(i) $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \forall x \in \mathbb{R}$

(ii) $x + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots \forall x \in \mathbb{R}$

(iii) $x - \frac{x^2}{2!} + \frac{x^3}{3!} - \dots \forall x \in \mathbb{R}$

(iv) $x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \forall x \in \mathbb{R}$

2. Write very short answer for each of the following questions:

1×6=6

(a) Define uniform continuity of a function on an interval.

(b) Use the sequential criterion for limit, find $\lim_{x \rightarrow 0} (x^2 \sin \frac{1}{x})$.

(c) Show that $\lim_{x \rightarrow 0} \frac{1}{x}$ does not exist in \mathbb{R} .

(d) Find the value of $\lim_{x \rightarrow \infty} \left(\frac{1}{x^5} + \tan^{-1} x^2 \right)$.

(e) State Caratheodory's theorem of derivatives.

(f) Define convex function.

3. Write short answer for each of the following questions: 3×5=15

(a) If $[x]$ denotes the greatest integer less than or equal to x , then discuss the kind of discontinuity at $x = 3$ for the function $f(x) = x - [x], \forall x \geq 0$.

(b) Show that the function $f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ is derivable at

$x = 0$.

(c) Verify Lagrange's Mean Value Theorem for the function

$f(x) = \sqrt{x^2 - 4}$ on $[2, 4]$.

(d) Examine the extreme values for the function

$f(x) = x^3 - 9x^2 + 15x + 7, x \in \mathbb{R}$.

(e) Let I be an interval, let x_0 be an interior point of I and let $n \geq 2$.

Suppose that the derivatives $f', f'', \dots, f^{(n)}$ exist and are continuous in a neighbourhood of x_0 and

$f'(x_0) = f''(x_0) = \dots = f^{(n-1)}(x_0) = 0$, but $f^{(n)}(x_0) \neq 0$. Prove

that f has a relative minimum at x_0 , if n is even and $f^{(n)}(x_0) > 0$

4. Write answer for each of the following questions:

(a) Using $(\varepsilon - \delta)$ definition of limit, show that

$$\lim_{x \rightarrow c} (x^2 + p) = c^2 + p.$$

(b) Prove that a function is uniformly continuous on an interval is continuous on that interval.

(c) If f is derivable at c and $f(c) \neq 0$, then prove that the function

$$\frac{1}{f} \text{ is also derivable thereat and } \left(\frac{1}{f} \right)' (c) = -\frac{f'(c)}{\{f(c)\}^2}.$$

(d) Find the relative maximum and relative minimum values of the function $f(x) = 2x^3 - 9x^2 - 24x$ (using first derivative test).

(e) Find the Taylor's series expansion of $\cos x$.

5. Answer any two of the following questions:

6×2=12

(a) Prove that a function f tends to a finite limit as x tends to c if and only if for every $\varepsilon > 0$, there exists a neighbourhood N of c such that $|f(x') - f(x'')| < \varepsilon \quad \forall x', x'' \in N$ and $x', x'' \neq c$.

(b) Show that the function $f(x)$ defined on \mathbf{R} by

$$f(x) = \begin{cases} x, & \text{when } x \text{ is rational} \\ -x, & \text{when } x \text{ is irrational} \end{cases} \text{ is continuous only at } x = 0.$$

(c) Prove that if f is a continuous function in $[a, b]$ and $f(a), f(b)$ are of opposite signs, then there exists a point $c \in]a, b[$ such that $f(c) = 0$.