DHANAMANJURI UNIVERSITY

Examination- 2024 (Dec)

Four year course B.Sc./B.A. 3^{rd} Semester

Name of Programme	**:**	**B.Sc./B.A. Mathematics**
Paper Type	**:**	**Theory**
Paper Code	**:**	**CMA-207**
Paper Title	**:**	**Theory of Real Funtions**

Full Marks : 80

Pass Marks : 32 **Duration: 3 Hours**

The figures in the margin indicate full marks for the questions:

Answer all the question.

1. Choose and rewrite the correct answer for each of the following questions: **$1 \times 3 = 3$**

a) Which of the following function is uniformly continuous on (0,1)?

i) $\frac{1}{x}$ ii) $\frac{sinx}{x}$ iii) $sin\frac{1}{x}$ iv) $\frac{cosx}{x}$

b) The function $f(x) = \begin{cases} x^p sin\frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ $, p \in \mathbf{N}$ is derivable at $x = 0$ only when

i) $p = 1$ ii) $p > 1$ iii) $p < 1$ iv) $p \geq 1$

c) The Taylor's series expansion of e^x is

i) $1 + x + \frac{x^2}{2} + \frac{x^3}{3} + ...$ ii) $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$

iii) $1 - x + \frac{x^2}{2} - \frac{x^3}{3} + ...$ iv) $x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$

2. Write very short answer for each of the following: $1 \times 6 = 6$

a) Define limit of a function ($\varepsilon - \delta$ approach).

b) Use the sequential criterion for limit, find $\lim_{x\to 2}(x^2+4x)$.

c) When is a function said to be continuous at a point?

d) Find the value of $\lim_{x\to\infty}\left(\frac{1}{x^2}+e^x\right)$.

e) State Lagrange's Mean Value Theorem.

f) Test the convexity of the function $f(x) = x + \frac{1}{x}$ for $x \in (0, \infty)$

3. Write short answer for each of the following questions: $3 \times 5 = 15$

a) Discuss the kind of discontinuity at $x = 0$, for the function

$$f(x) = \begin{cases} \frac{\sin x}{x}, & \text{when } x \neq 0 \\ 0, & \text{when } x = 0 \end{cases}$$

b) Show that the function $f(x) = x^2$ is derivable on [0,1].

c) Verify the Rolle's Theorem for the function $f(x) = (x-a)^m(x-b)^n$, where m and n are positive integers on $[a, b]$.

d) Examine the extreme values of the function $f(x) = x^3 - 6x^2 + 9x + 3, x \in R$

e) Use Taylor's Theorem to show that $\cos x \geq 1 - \frac{x^2}{2}$, for all real x.

4. Write answer for each of the following questions: $4 \times 5 = 20$

a) If f and g are two functions defined on some neighbourhood of c such that $\lim_{x\to c} f(x) = l, \lim_{x\to c} g(x) = m$, then prove that $\lim_{x\to c}(f.g)(x) = lm$.

b) Evaluate $\lim_{x \to 0} e^x \text{ sgn } (x + [x])$, where the signum function is defined as

$$\text{sgn}(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x < 0 \end{cases}$$

and $[x]$ means the greatest integer less than or equal to x.

c) Show that the function $f(x) = |x|$ is continuous but not derivable at the origin.

d) Let c be an interior point of the interval I at which $f : I \to R$ has a relative extremum. If the derivative of f at c exists, then prove that $f'(c) = 0$.

e) Write the Taylor's series expansion of sin x.

5. Answer any two of the following questions: $6 \times 2 = 12$

a) If a function f is continuous on $[a, b]$, then prove that it attains its bounds at least once in $[a, b]$.

b) Using $\epsilon - \delta$ definition of limit , show that $\lim_{x \to 2} \frac{x^3 - 4}{x^2 + 1} = \frac{4}{5}$.

c) Prove that a function which is continuous on a closed interval is also uniformly continuous on that interval.

6. Answer any two of the following questions: $6 \times 2 = 12$

a) Let f be defined on an interval I containing the point c. Prove that the function f is differentiable at c if and only if there exista a function ϕ on I that is continuous at c and satisfies
$f(x) - f(c) = \phi(x)(x - c)$ for $x \in I$.

b) Show that $\frac{v - u}{1 + v^2} < tan^{-1}v - tan^{-1}u < \frac{v - u}{1 + u^2}$ if $0 < u < v$ and deduce that $\frac{\pi}{4} + \frac{3}{25} < tan^{-1}\frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}$.

c) State and prove Darboux's Theorem of differentiability.

7. Answer any two of the following questions: **6 × 2 = 12**

a) State and prove Taylor's Theorem.

b) Let I be an open interval and the function $f : I \to R$ have a second derivative on I. Prove that f is a convex function on I if and only if $f''(x) \geq 0, \forall x \in I$.

c) Approximate the function $f(x) = \sqrt[3]{1+x}, x > -1$ by a polynomial of degree 2 and also give the error in terms of x.

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